OPINION DYNAMICS WITH BOUNDED CONFIDENCE IN THE BAYES RISK ERROR DIVERGENCE SENSE

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ABSTRACT

Bounded confidence opinion dynamic models have received much recent interest as models of information propagation in social networks and localized distributed averaging. However in the existing literature, opinions are only viewed as abstract quantities rather than as part of a decision-making system. In this work, opinion dynamics are examined when agents are Bayesian decision makers that perform hypothesis testing or signal detection. Bounded confidence is defined on prior probabilities of hypotheses through Bayes risk error divergence, the appropriate measure between priors in hypothesis testing. This definition contrasts with the measure used between opinions in the standard model: absolute error. It is shown that the rapid convergence of prior probabilities to a small number of limiting values is similar to that seen in the standard model. The most interesting finding in this work is that the number of these limiting values changes with the signal-to-noise ratio in the hypothesis testing task. The number of final values or clusters is maximal at intermediate signal-to-noise ratios, suggesting that the most contentious issues lead to the largest number of factions.

Index Terms— Bayesian decision making, Bregman divergence, Krause model, opinion dynamics, signal detection, social network

1. INTRODUCTION

People are often asked for their opinion. Will *Man of Steel* be a blockbuster or a flop? Who will win the presidential election? Will Oracle Database or IBM DB2 have more sales next year? Are bomber jackets going to be popular this spring?

In all of these questions, one of the hypotheses is true and the other one is false after the fact (assuming consistent definitions of blockbuster, flop, win, and popular). The job of the opinion giver is to try to report the true hypothesis as his or her opinion. Thus, such opinion questions can be framed as Bayesian binary hypothesis testing or signal detection problems because human decision makers are tasked with predicting the true hypothesis based on noisy observations of the world and their prior belief with minimal probability of error or Bayes risk [1]. Under such a model, an opinion is composed of both an observation and a prior.

The observation component of opinion can have varying levels of noise. For some opinions, observable features are very predictive of the hypothesis whereas in others, the observable features are not very predictive at all. In society, the prior beliefs of decision makers are mutable and influenced by their network of acquaintances. People sway others one way or the other. However, most people can only be swayed a little bit at a time; they are not influenced by people with a very different prior. This non-influence of people with beliefs too far away is known as bounded confidence [2].

One may ask why signal detection questions rather than more direct consumer choice questions such as whether the decision maker will personally buy a bomber jacket or who the decision maker will vote for are of interest. One reason is that it has been empirically shown that aggregating answers from the former type of question (expectation question) yields more accurate forecasts than the latter type of question (intention question) [3]. Also, the former type of question is more in line with the popular definition of opinion.

Opinion dynamics — how opinions evolve among the people in a social network over time — has been studied in a variety of literatures recently [4]. Two discrete-time models of note that incorporate the idea of bounded confidence are the Krause model and the Deffuant-Weisbuch model [2, 5, 6]. The Krause model in particular begins with an initial assignment of opinions for each person; for all people in the society, it works by finding all others whose opinion is within a certain absolute value of ones own and updating the own opinion to be the mean of those opinions. The model has been analyzed to show that opinions converge in finite time, usually quickly, to a few values [7], resulting in clusters of people with the same final opinion.

The existing literature on bounded confidence opinion dynamics only considers opinions as abstract real-valued numbers without connection to a decision-making formulation. In this work, it is proposed that opinion-giving be treated as a Bayesian signal detection task and that bounded confidence dynamics be applied to the decision makers' prior probabilities. Such a view of opinion dynamics through a decision-theoretic lens seems not to exist in the literature.

When one is to quantify dissimilarity in prior probabilities of Bayesian hypothesis testing, absolute error is not appropriate because it treats all prior probabilities the same. The difference between a prior probability of 0.1 and 0.2 leads to a Bayes risk difference that may be profoundly different than the Bayes risk difference between a prior probability of 0.4 and 0.5, which is also different when there are different observation models and signal-to-noise ratios. Bayes risk error divergence, a member of the family of Bregman divergences, is a criterion by which the dissimilarity of two input prior probabilities is examined appropriately according to detection performance [8, 9]. Thus, in this work, it is proposed that Bayes risk error be used to define bounded confidence.

Under bounded confidence defined in the sense of Bayes risk error divergence, it is shown in this paper that prior probabilities converge in finite time and quickly to a few limiting values in the same way that opinions converge in the standard Krause model with bounded confidence defined by absolute error. Much more interestingly, by defining opinion dynamics in the decision-making setting, it is possible to explore the effect of measurement noise on the convergence. It is found that the largest number of converged clusters occurs for an intermediate signal-to-noise ratio whereas convergence is to only one value for very low and very high signal-to-noise ratios. An intermediate signal-to-noise ratio corresponds to an issue that is quite contentious. Thus, this novel finding in this paper relays the interpretation that more contentious issues lead to more factions.

The remainder of the paper is organized as follows. First, some background material is covered on the Bayes risk error divergence in Section 2 and on the Krause model of bounded confidence opinion dynamics in Section 3. Then in Section 4, the new model for opinions when viewed as prior probabilities for decision-making tasks is proposed by considering bounded confidence in the Bayes risk error divergence sense. Section 5 proves convergence of the decision weights of decision makers in finite time to clusters. Empirical results are given in Section 6 that show the dynamics of the proposed model, including the varying number of clusters as a function of signal-to-noise ratio in the Bayesian hypothesis testing task. Section 7 discusses and contextualizes the results and offers directions for future work.

2. BAYESIAN HYPOTHESIS TESTING AND BAYES RISK ERROR

Let us consider the standard Bayesian approach to signal detection or hypothesis testing, focusing on the binary hypothesis case [1]. There is a binary hypothesis $H \in \{h_0, h_1\}$ with prior probabilities p_0 and p_1, both non-negative such that $p_0+p_1 = 1$. The opinion giver or decision maker does not observe the hypothesis directly, but observes a noisy measurement Y that is related to the hypothesis through the likelihood functions $f_{Y|H}(y|H = h_0)$ and $f_{Y|H}(y|H = h_1)$. The decision maker observes y and tries to determine whether the true hypothesis is h_0 or h_1 using the decision rule $\hat{h}(y)$.

The decision maker has error costs c_{10} and c_{01}, the costs of deciding h_1 when the true hypothesis is h_0 (a Type I error) and deciding h_0 when the true hypothesis is h_1 (a Type II error). The decision rule that is considered is the following likelihood ratio test:

$$\frac{f_{Y|H}(y|H = h_1)}{f_{Y|H}(y|H = h_0)} \underset{\hat{h}(y)=h_0}{\overset{\hat{h}(y)=h_1}{\gtrless}} \frac{ac_{10}}{(1-a)c_{01}}, \tag{1}$$

where if the decision weight $a \in [0, 1]$ equals p_0, the decision rule is Bayes optimal. The resulting type I error probability of the decision rule, denoted p_E^{I}, is a function of the decision weight a in the threshold of the likelihood ratio test, i.e. $p_E^{\mathrm{I}}(a)$. This is similarly the case for the type II error probability $p_E^{\mathrm{II}}(a)$. The overall detection error probability weighted by the costs and the true priors, known as the Bayes risk, is:

$$J(p_0, a) = c_{10}p_0 p_E^{\mathrm{I}}(a) + c_{01}(1-p_0)p_E^{\mathrm{II}}(a). \tag{2}$$

The difference between the Bayes risk of the likelihood ratio test with decision weight $a = p_0$ and the Bayes risk of the likelihood ratio test with any general decision weight a is defined to be the Bayes risk error divergence [8, 9]:

$$d(p_0, a) = J(p_0, a) - J(p_0, p_0). \tag{3}$$

The Bayes risk error divergence is the appropriate measure of proximity between prior probabilities of hypotheses because it is directly related to decision-making performance.

3. KRAUSE MODEL OF BOUNDED CONFIDENCE OPINION DYNAMICS

Let us also consider a now standard approach to modeling opinion dynamics: the Krause model [2, 5]. We have a society of opinion givers $i = 1, \ldots, n$, each with an opinion at time k of $a_i(k) \in \mathbb{R}$. Each opinion giver has a neighborhood of other opinion givers in the society whose opinion does not differ by more than an opinion threshold τ:

$$N_i(k) = \{j \in \{1, \ldots, n\} \mid |a_i(k) - a_j(k)| \le \tau\}. \tag{4}$$

The neighborhood defined by the threshold τ is the model for bounded confidence.

Each opinion giver begins with an initial opinion. The opinions are updated based on interaction and influence from neighbors. The opinion giver's opinion is updated at the next time step to the mean or centroid of the neighborhood at the current time step:

$$a_i(k+1) = \frac{1}{|N_i(k)|} \sum_{j \in N_i(k)} a_j(k). \tag{5}$$

It has been shown that these dynamics converge in a finite number of steps to a few clusters with all opinion givers in each cluster sharing the same opinion value [7].

4. BAYES RISK ERROR BOUNDED CONFIDENCE

We now propose a bounded confidence opinion dynamics model for Bayesian decision makers by endowing opinion-giving with a decision-making, specifically detection, task. We do so by combining the ideas discussed in Section 2 and Section 3, specifically by having opinion givers observe a noisy measurement Y and use the likelihood ratio test decision rule (1), and have their personal prior beliefs evolve according to Krause dynamics.

The key difference in the dynamics from the standard Krause model is in the quantification of proximity for bounded confidence. When opinions have a semantic meaning as prior probabilities in a detection task, absolute error as in (4) is not how similarity between opinions should be measured. As discussed earlier, the divergence between prior probabilities that takes detection performance into account is the Bayes risk error. Thus, we take the bounded confidence neighborhood to be defined by Bayes risk error divergence:

$$N_i(k) = \{j \in \{1, \ldots, n\} \mid d(a_i(k), a_j(k)) \le \tau\}. \tag{6}$$

It should be noted that due to the likelihood functions in the decision rule and consequently Bayes risk, the Bayes risk error divergence changes with different amounts of observation noise.

The second part of the Krause dynamics is the opinion update to the centroid of the neighborhood. The Bayes risk error divergence centroid of a set of prior probabilities is simply the mean value due to a property of the family of Bregman divergences [10], of which the Bayes risk error divergence is a member. Therefore, the opinion update (5) remains unchanged from the standard Krause model when Bayes risk error is considered.

5. CONVERGENCE RESULTS

In this section, we present basic convergence properties of the opinion dynamics system with bounded confidence defined in the Bayes risk error sense. The first proposition states that the initial ordering of the decision weights is preserved throughout the dynamic evolution. The second proposition states that the dynamics of the decision weights converges in finite time to a few clusters. The propositions and proof techniques are very similar to those for the standard Krause model presented in [7]. Due to space constraints, we omit the full proof of the second proposition.

Proposition 1. *Let* $\{a_1(k), \ldots, a_n(k)\}$ *evolve according to the dynamics* (6), (5). *If* $a_i(0) \leq a_j(0)$, *then* $a_i(k) \leq a_j(k)$ *for all* k.

Proof. The proof is by induction. The base case is in the theorem statement: $a_i(0) \leq a_j(0)$. We assume that $a_i(\kappa) \leq a_j(\kappa)$ and show that $a_i(\kappa+1) \leq a_j(\kappa+1)$ must then also be true.

Let

$$S_i(\kappa) = N_i(\kappa) \setminus N_j(\kappa),$$
$$S_j(\kappa) = N_j(\kappa) \setminus N_i(\kappa),$$
$$S_{ij}(\kappa) = N_i(\kappa) \cap N_j(\kappa),$$

and also let

$$\bar{a}_{S_i}(\kappa) = \frac{1}{|S_i(\kappa)|} \sum_{\ell \in S_i(\kappa)} a_\ell(\kappa),$$
$$\bar{a}_{S_j}(\kappa) = \frac{1}{|S_j(\kappa)|} \sum_{\ell \in S_j(\kappa)} a_\ell(\kappa),$$
$$\bar{a}_{S_{ij}}(\kappa) = \frac{1}{|S_{ij}(\kappa)|} \sum_{\ell \in S_{ij}(\kappa)} a_\ell(\kappa).$$

Since Bayes risk error is a Bregman divergence, although not symmetric and not convex in the second argument like absolute error, it is monotonically non-decreasing in the second argument. This monotonicity leads to the conclusion with neighborhoods defined by Bayes risk error divergence that for any $\ell_1 \in S_i(\kappa)$, $\ell_2 \in S_{ij}(\kappa)$, and $\ell_3 \in S_j(\kappa)$, $a_{\ell_1}(\kappa) \leq a_{\ell_2}(\kappa) \leq a_{\ell_3}(\kappa)$. Therefore, it is also true that $\bar{a}_{S_i}(\kappa) \leq \bar{a}_{S_{ij}}(\kappa) \leq \bar{a}_{S_j}(\kappa)$.

Carrying out the update (5), we find that

$$a_i(\kappa+1) = \frac{|S_i(\kappa)|\bar{a}_{S_i}(\kappa) + |S_{ij}(\kappa)|\bar{a}_{S_{ij}}(\kappa)}{|N_i(\kappa)|} \leq \bar{a}_{S_{ij}}(\kappa),$$
$$a_j(\kappa+1) = \frac{|S_j(\kappa)|\bar{a}_{S_j}(\kappa) + |S_{ij}(\kappa)|\bar{a}_{S_{ij}}(\kappa)}{|N_j(\kappa)|} \geq \bar{a}_{S_{ij}}(\kappa),$$

so $a_i(\kappa+1) \leq a_j(\kappa+1)$. □

Proposition 2. *Let* $\{a_1(k), \ldots, a_n(k)\}$ *evolve according to the dynamics* (6), (5). *For all* i, $a_i(k)$ *converges to a limit* a_i^* *in finite time. Additionally, for any pair* i *and* j, *either* $a_i^* = a_j^*$ *or* $max\{d(a_i^*, a_j^*), d(a_j^*, a_i^*)\} \geq \tau$.

Proof. The proof follows from Proposition 1 and the bounded sequence convergence argument for proving [7, Theorem 1], with distance replaced by the maximum of the divergences with both orderings of input arguments. □

6. EMPIRICAL RESULTS

In this section, we present examples to illustrate opinion dynamics with bounded confidence in the Bayes risk error divergence sense. We consider a set of $n = 100$ opinion givers with initial prior probabilities that are samples from a uniform distribution over the interval $[0, 1]$. The signal detection task is composed of hypotheses and noise such that conditioned on hypothesis h_0, y is drawn from a scalar Gaussian distribution with mean 0 and standard deviation σ. Conditioned on hypothesis h_1, y is also Gaussian with standard deviation σ, but with mean 1. All opinion givers have observations of the same quality; thus these likelihood functions are the same for all 100 opinion givers. All opinion givers also have the same costs c_{10} and c_{01} which are equal. The opinion threshold is fixed at $\tau = 0.1$.

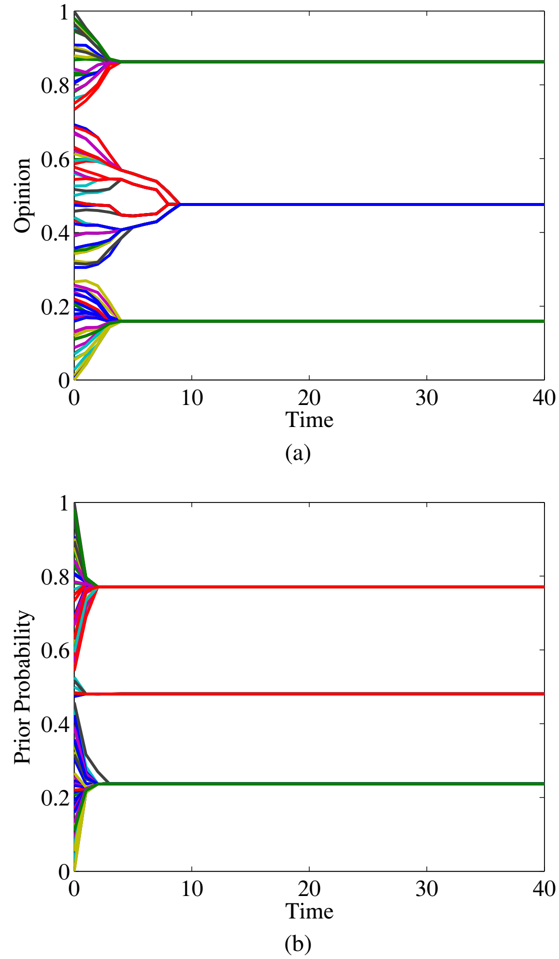

Fig. 1. Dynamics of 100 people with bounded confidence threshold 0.1 for (a) standard Krause model and (b) Bayes risk error divergence Krause model with univariate Gaussian observations having means 0 and 1, and standard deviation 4.

As a point of comparison, we also show opinion dynamics under the standard Krause model.

Fig. 1(a) shows the standard Krause opinion dynamics. Fig. 1(b) shows the dynamics with neighborhoods defined by Bayes risk error as proposed in this paper with a particular value for the observation noise standard deviation $\sigma = 4$. The convergence behavior under Bayes risk error bounded confidence and standard absolute error bounded confidence is qualitatively similar. Both converge quickly to a few clusters while maintaining the initial ordering. Under Bayes risk error, the convergence is faster and the middle cluster contains fewer people. Most people end up on one side of the fence or the other rather than straddling it.

Since (unlike absolute error) the Bayes risk error divergence depends on the likelihood functions of the observation model, bounded confidence opinion dynamics can be investigated under different signal-to-noise ratios. For smaller and greater noise levels, as seen in Fig. 2, the convergence pattern as well as the number of final clusters is different than an intermediate amount of noise.

The number of converged clusters (averaged across several trials

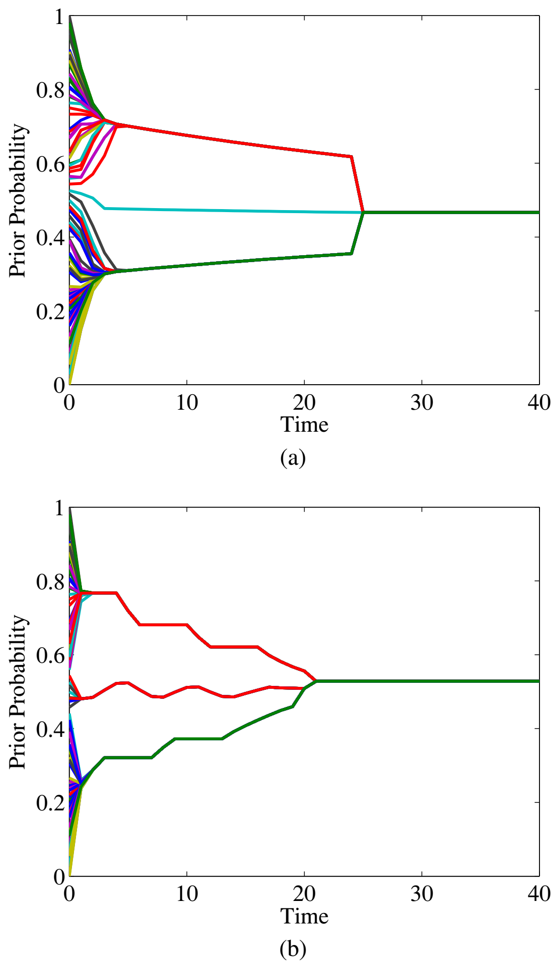

Fig. 2. Dynamics of 100 people with bounded confidence threshold 0.1 for Bayes risk error divergence Krause model with univariate Gaussian observations having means 0 and 1, and (a) standard deviation 1 and (b) standard deviation 16.

of initial decision maker prior probabilities) as a function of noise level for a fixed bounded confidence threshold is shown in Fig. 3. The number of clusters first increases and then decreases with the noise level. When observations are either very certain or very uncertain, there is convergence to a single cluster; however, in the intermediate case there are more clusters. Such a phenomenon with the number of converged clusters in bounded confidence opinion dynamics does not appear in any previous work.

7. DISCUSSION AND CONCLUSION

In this paper, we have examined bounded confidence opinion dynamics when the opinions are endowed with a meaning as decisions in a hypothesis testing or signal detection task. The model that arises in this setting is very similar to the standard Krause model of opinion dynamics, but with the key difference that the confidence bound is calcuated via Bayes risk error divergence rather than absolute error. The convergence results and behaviors in this setting follow those of the Krause model. By being endowed with the decision-making

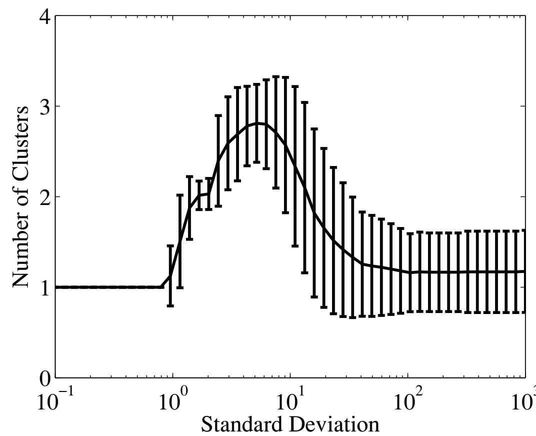

Fig. 3. Converged number of clusters of 100 people with bounded confidence threshold 0.1, averaged over 200 trials, for Bayes risk error divergence Krause model with univariate Gaussian observations having means 0 and 1, and standard deviation as plotted. The error bars indicate the standard deviation of the number of clusters over the trials. The converged number of clusters is one in all trials for small noise standard deviation.

context, we are able to examine the behavior of the opinion dynamics as a function of observation noise. In doing so, we find that the number of converged clusters first increases and then decreases with signal-to-noise ratio.

The interpretation is that at very low noise, the observation readily indicates the true hypothesis so priors do not matter, and thus everyone converges to an average, maximally ignorant prior. The intermediate noise regime corresponds to the most contentious issues and opinions because the observation is somewhat informative of the true hypothesis, but not fully. The choice of prior has a large effect on the opinion, and thus these contentious issues lead to factions. With very large amounts of noise, no one has any idea what is going on, and everyone is uninformed; again, convergence is to an average, uninformed consensus of priors. Although such a phenomenon is known to occur in society, to the best of the author's knowledge, there has been no previously proposed opinion dynamics model that generates more factions when the issue is more contentious.

The theoretical convergence analysis presented herein, and also in the literature for the standard Krause model has not been able to precisely characterize the number of converged clusters, which is central to the main insight of this paper; theoretical analysis of that characterization is of great importance. Other directions for further research include examining a more realistic version of the model in which only other people with whom a social connection exists are included in the opinion update mean [11], including stubborn agents [12], considering agents with differing Bayes costs [13], and studying vector-valued opinions or prior probabilities [14] arising when voting on several semantically-related issues [15], which are readily handled with Bayes risk error divergence [8].

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